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Basic embeddings into a product of graphs

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Abstract

The notion of a basic embedding appeared in research motivated by Kolmogorov–Arnold's solution of Hilbert's 13th problem. Let K, X, Y be topological spaces. An embedding $K \subset X \times Y$ is called *basic* if for every continuous function $f: K \to \mathbb{R}$ there exist continuous functions $g: X \to \mathbb{R}$, $h: Y \to \mathbb{R}$ such that $f(x, y) = g(x) + h(y)$ for any point $(x, y) \in K$. Let T_i be an i-od.

Theorem. *There exists only a finite number of 'prohibited' subgraphs for basic embeddings into $\mathbb{R} \times T_n$. Consequently, for a finite graph K there is an algorithm for checking whether K is basically embeddable into $\mathbb{R} \times T_n$.* Our theorem is a generalization of Skopenkov's description of graphs basically embeddable into $\mathbb{R}^2$, and our proofs is a (non-trivial) extension of that one. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Hilbert conjectured in his 13th problem that there are continuous functions of three variables which are not representable as a composition of continuous functions of two variables. Arnold and Kolmogorov proved in [2,4] that every continuous function of several variables defined on a compact subset of $\mathbb{R}^2$ admits a representation as a sum of $2n + 1$ continuous functions of one variable.

Let X, K, Y be topological spaces. An embedding $K \subset X \times Y$ is called *basic* (and denoted by $K \subset_b X \times Y$) if for every continuous function $f: K \to \mathbb{R}$ there exist continuous functions $g: X \to \mathbb{R}$, $h: Y \to \mathbb{R}$ such that $f(x, y) = g(x) + h(y)$ for any point $(x, y) \in K$. This condition can be reformulated in terms of function spaces as follows [10]. Given a

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map $\phi : K \to X \times Y$, consider ϕ as a product of two maps $\alpha : K \to X$ and $\beta : K \to Y$. Let the linear superposition operator $\Phi : C(X) \oplus C(Y) \to C(K)$ be given by

$$\Phi(g,h)(x) = g\bigl(\alpha(x)\bigr) + h\bigl(\beta(x)\bigr).$$

Then an embedding is basic if and only if Φ maps $C(X) \oplus C(Y)$ *onto* $C(K)$.

The weaker version of Arnold–Kolmogorov's theorem is that the n-dimensional cube is basically embeddable in $\mathbb{R}^{2n+1}$. The following theorem describing the compacta basically embeddable in $\mathbb{R}^m$ for $m \geqslant 3$ is proved in [6] and [9]: a compactum X is basically embeddable in $\mathbb{R}^m$ if and only if $\dim X \leqslant (m-1)/2$. Trivially, X is basically embeddable in $\mathbb{R}$ if and only if X is topologically embeddable there. The description of pathwise-connected compacta basically embeddable in $\mathbb{R}^2$ in terms of prohibited subcontinua is given in [8]. In a partial, case there are characterizations of finite graphs basically embeddable in $\mathbb{R}^2$ in terms of prohibited subgraphs and universal trees in [3, Theorem 1.2]. We can reformulate these criteria as follows: "A finite graph K is basically embeddable into $\mathbb{R}^2$ if and only if K has no bad vertices (or, equivalently, $\delta(K) = 0$)" (see necessary definitions below). But the general problem of characterizing the compacta basically embeddable in $\mathbb{R}^2$ is still open.

Basic embeddings into a product of dendrites were studied in [10, Theorem 4.6, p. 29]. Let T_i be an i-od (or a star with i rays). The purpose of this paper is to describe finite graphs basically embeddable into $\mathbb{R} \times T_n$. Moreover we obtain some necessary and sufficient conditions for basic embeddability of graphs into $T_m \times T_n$ for $m \geqslant 3$. This is a solution of some problems from the preliminary version of [3].

Let us make some necessary definitions. Call a vertex (i.e., either an endpoint or a branched point) of a finite graph K *horrid* if its degree is greater than 4. Call a vertex of K *awful* if its degree equals 4 and it has no hanging edges. Call a vertex of K *bad* if it is either awful of horrid. Call a bad vertex of K *dry* if it has a hanging edge. Clearly, a dry vertex is a horrid vertex. The *defect* of K is the sum $\delta(K) = (\deg A_1 - 2) + \cdots + (\deg A_k - 2)$, where $A_1, \ldots, A_k$ are the bad vertices of K. Further we suppose $n \geqslant 3$.

Theorem 1.1. *A finite* (*not necessarily connected*) *graph K is basically embeddable into $\mathbb{R} \times T_n$ if and only if K is a tree and either $\delta(K) < n$ or $\delta(K) = n$ and K has a dry vertex.*

Corollary 1.2. *A finite graph K is basically embeddable into $\mathbb{R} \times T_3$* (*or, equivalently, $T_2 \times T_3$*) *if and only if either of the two following equivalent conditions holds*:

(a) (*cf.* [5]) *K does not contain any of the graphs of Fig.* 1;

(b) *K is contained in W_n for some n* (*see Fig.* 2).

Now we shall construct universal graphs W_n for basic embeddings into $\mathbb{R} \times T_3$. Let U_1 be T_3, A a hanging edge of U_1 and a the hanging endpoint of A. The graph U_{n+1} is obtained from U_n by branching every hanging edge except A. Let V_n be the graph obtained by gluing one hanging edge to every non-hanging vertex of U_n. The vertex a is called the *root* of U_n and V_n. Let W_n be the wedge of four copies of V_n and an arc such that the roots of V_n attach to one endpoint of the arc.

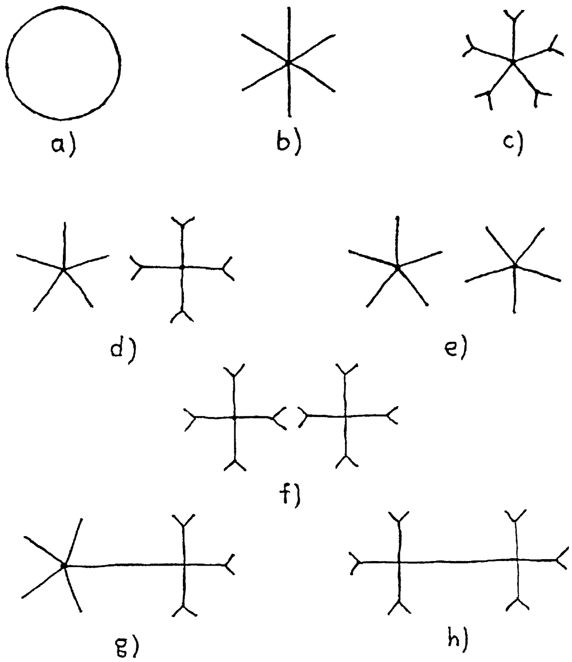

Fig. 1.

Corollary 1.3. *There exists only a finite number of 'prohibited' subgraphs for basic embeddings into $\mathbb{R} \times T_n$. Consequently, for a finite graph K there is an algorithm for checking whether K is basically embeddable into $\mathbb{R} \times T_n$.*

Theorem 1.4. *If a finite* (*not necessarily connected*) *graph K is basically embeddable into $T_m \times T_n$ $(m \geqslant 3)$, then K is a tree and one of the two following conditions holds*:

(1.4.1) *either $\delta(K) < m+n-2$, or $\delta(K) = m+n-2$ and K has a dry vertex*;

(1.4.2) *all bad vertices of K are split into two collections $a_1, \ldots, a_k$ and $b_1, \ldots, b_l$ such that*

$$(\deg a_1 - 2) + \cdots + (\deg a_k - 2) \leqslant n,$$

$$(\deg b_1 - 2) + \cdots + (\deg b_l - 2) \leqslant m.$$

Moreover, if the first (*second*) *weak inequality is equality, then a_1* (*b_1, respectively*) *is dry. In particular, $\delta(K) \leqslant m+n$.*

If condition (1.4.1) *holds $m \geqslant 2$, then K is basically embeddable into $T_m \times T_n$.*

The proof of Theorems 1.1 and 1.4 is based on the reduction of the property of being a basic embedding to a pure geometric condition [10, Lemma 2.23(iii), p. 14], and on an extension of techniques from [8]. It seems that Theorem 1.4 is unnaturally more complicated than Theorem 1.1. But there is the following graph K basically embeddable into $T_3 \times T_3$, for which (1.4.1) does not hold. Let K be a disjoint union of two pentods, i.e., $\delta(K) = 6$. Fix a hanging edge C (D) in a triod T_3 (T_3') with the center c (d, respectively).

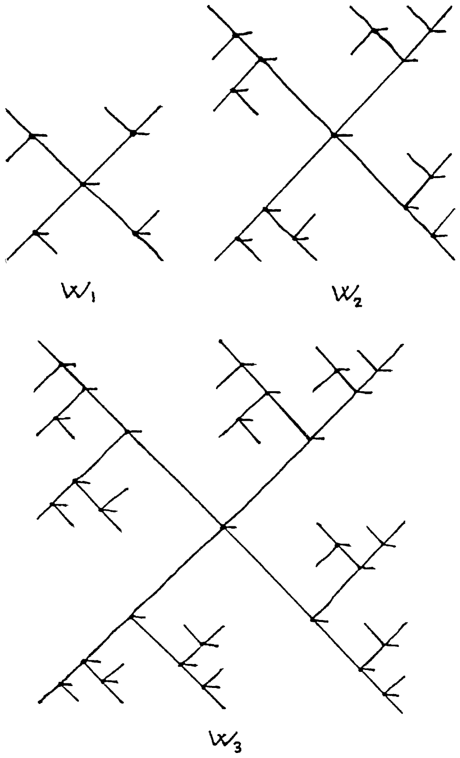

Fig. 2.

Then the subset $C \times T_3' \cup T_3 \times D \subset T_3 \times T_3'$ consists of two 'books' with three 'pages' pasting together. Basically embed each pentod into its 'book', as in Corollary 1.2, and such that its projections on $c \times D$ and $C \times d$ are mutually disjoint. Then we have a basic embedding $K \subset T_3 \times T_3'$. It should be quite trivial after the reading of Section 5. This example shows an essential difference between the cases $m = 2$ and $m > 2$. The paper is organized as follows. In Section 2 we introduce the main tools of studying basic embeddings and prove some easy lemmas. In Section 3 we prove necessity in Theorems 1.1 and 1.4 using these lemmas. We split the proof of sufficiency in Theorems 1.1 and 1.4 into three parts. The first part is a description of an admissible tree (Section 4). The second part is the basic embeddability of an admissible tree (Theorem 5.1 in Section 5). The third part is the proof that each connected tree satisfying condition (1.4.1) is an admissible tree (Theorem 6.1 in Section 6). Since the conditions of Theorem 1.1 are the partial case of (1.4.1) (for $m = 2$), then sufficiency in Theorems 1.1 and 1.4 will be proved. Thus, we can formulate a criteria for basic embeddings into $\mathbb{R} \times T_n$ as follows: "A finite connected graph K is basically embeddable into $\mathbb{R} \times T_n$ if and only if K is an admissible tree". In Section 7 we prove Corollaries 1.2 and 1.3. In Section 8 we formulate some interesting conjectures for basic embeddings into a product of finite graphs. All constructions in the paper are simplified for basic embeddings into $\mathbb{R} \times T_n$. At the beginning of Sections 3–6 we make some remarks for this partial case.

2. Preliminaries

Let X, Y be finite graphs. By p_x and p_y we denote the projections $p_x : X \times Y \to X$, $p_y : X \times Y \to Y$. For $Z \subset X \times Y$ let

$$E(Z) = \bigl\{z \in Z\colon \operatorname{card}\bigl(Z \cap (p_x z \times Y)\bigr) > 1 \text{ and } \operatorname{card}\bigl(Z \cap (X \times p_y z)\bigr) > 1\bigr\}.$$

A sequence $\{a_1, \ldots, a_n\} \subset X \times Y$ is called an *array*, if for each i, $a_i \neq a_{i+1}$, and $p_x(a_i) = p_x(a_{i+1})$ for odd i and $p_y(a_i) = p_y(a_{i+1})$ for even i. The proof of [10, Lemma 2.23(iii), p. 14], [8, GC, p. 33] holds for a more general case:

GC 2.1. *An embedding $K \subset X \times Y$ is not basic if and only if*
(2.1.1) *$E^n(K) \neq \emptyset$ for each n, or*
(2.1.2) *for each n there exists an array of n points in K.*

By c and d we denote the centers of T_m and T_n, respectively.

Basic non-embeddability of S into $T_m \times T_n$ (cf. [10, proof of Proposition 2.21, p. 15]). Suppose to the contrary that $S \subset_b T_m \times T_n$. Since S is a finite graph, then $p_x S$ $(p_y S)$ either is a join of at most m (n) arcs, containing the vertex c $(d$, respectively) or is an arc. Evidently, for any point $a \in \operatorname{Int} p_x S$ $(\operatorname{Int} p_y S)$ we have that $(a \times T_n) \cap S$ $((T_m \times a) \cap S$, respectively) consists of more than one point. Hence $S - E(S)$ consists of at most $m + n$ points. A simple inductive argument shows that for each $i > 0$, $E^i(S)$ is a cofinal set in S, and in particular is nonempty, contradicting GC 2.1.

An arc A is called *horizontal* (*vertical*) if $p_y A$ $(p_x A$, respectively) is a point. An arc is called a *compression* arc if it is either horizontal or vertical.

Definition 2.2. Suppose that $K \subset X \times Y$ and $I \subset K$ $(J \subset K)$ is a horizontal (vertical, respectively) arc. A compression generated by I, J is the map

$$q = (r \times \operatorname{id} Y) \circ (\operatorname{id} X \times s) : X \times Y \to (X/p_x I) \times (Y/p_y J),$$

where $r : X \to X/p_x I$ and $s : Y \to Y/p_y J$ are the projections.

Compression Lemma 2.3. *Let K, X, Y be finite graphs, $K \subset_b X \times Y$ and I, J and q be as above. Then*
(2.3.1) $qK \subset_b (X/p_x I) \times (Y/p_y J)$;
(2.3.2) $q|_{K-(I\cup J)}$ *is a homeomorphism.*

Proof. The proof of (2.3.1) and (2.3.2) is analogous to [8, §2, "proof of Compression Lemma"]. With the following alterations: "the segment $[a, b]$ is parallel to x-coordinate (y-coordinate) axis" to "$[a, b]$ is a horizontal (vertical, respectively) arc", and 'arc I orthogonal to arc J' to 'either both $p_x I$ and $p_y J$ or both $p_x J$ and $p_y I$ are points'. $\square$

3. Proof of necessity in Theorems 1.1 and 1.4

The structure of the proof is as follows. See Diagram 1. Necessity in Theorems 1.1 and 1.4 in the simple case (when all awful vertices of K lie in Γ) follows from (3.1.1) and (3.1.2) in Proposition 3.1. The general case follows from the simple one, Reduction Lemma 3.2 and Compression Lemma 2.3. We prove (3.1.1) and (3.1.3) analogously using Induction Lemma 3.3. We prove (3.1.2) and Reduction Lemma 3.2 analogously using (3.1.3). In Proposition 3.1 we shall consider basic embeddings of a finite tree K into $G \times H$, where G and H are subpolyhedra of T_m and T_n, respectively, and such that some products of hanging vertices of G and H correspond to some non-hanging vertices of K.

Our proof is based on two ideas. The first idea is used in (3.1.1) and (3.1.3), which are generalizations of [8, "Basic non-embeddability of C_4"] and [8, "the cross lemma"], respectively. The second idea is used in (3.1.2) and Reduction Lemma 3.2, which are generalizations of [8, "Basic non-embeddability of C_4"]. So, before reading the proofs below it will be helpful to look at the corresponding proofs in [8].

By Γ denote the singular set of $T_m \times T_n$. Evidently, $\Gamma = c \times T_n \cup T_m \times d$ for $m, n \geqslant 3$ and $\Gamma = c \times T_n$ for $m = 2$, $n \geqslant 3$. Finally, Γ is a graph. Consider $T_m \times T_n$ as the union of $I \times J$, where $I \subset T_m$, $J \subset T_n$ are 'rays', i.e., arcs, with ends c and d. From [8, Theorem 1] follows that all horrid vertices of K lie in Γ (actually, no neighborhood of a horrid vertex in K can be basically embeddable into $I \times J$).

Definition (G, T_j-structure on $\mathbb{R} \times T_n$). Let $G \subset \mathbb{R}$ be a disjoint union of arcs and $H = T_j \subset T_n$ be a substar. Let $g_1, \ldots, g_s$ be arbitrary distinct points of G. Then $(G, T_j, \{g_i\})$ is called a G, T_j-structure on $\mathbb{R} \times T_n$. Let $M(G, T_j, \{g_i\})$ be the sum of j and degrees of points $g_1, \ldots, g_s$ in G.

Evidently, each point g_i has degree 2 in $\mathbb{R}$. Hence $M(\mathbb{R}, T_j, \{g_i\}) = j + 2s$. For the necessity in Theorem 1.1 we may omit cases 2, 3 below. And also in Induction Lemma 3.3,

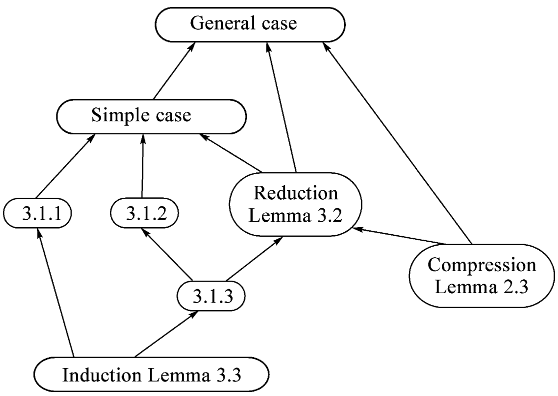

Diagram 1.

if $H = T_j$, then $H' = H - p_y \mathring{B} = T_{j-1}$. A G, T_j-structure on $\mathbb{R} \times T_n$ is the partial case of a G, H-structure $(G, H, \{g_i\}, \{h_j\})$ on $T_m \times T_n$, where $H = T_j$, $\{h_j\} = \emptyset$.

Definition (G, H-structure on $T_m \times T_n$). Let $G \subset T_m$, $H \subset T_n$ be subpolyhedra containing c, d, respectively. Let $g_1, \ldots, g_s$ $(h_1, \ldots, h_t)$ be arbitrary distinct points of G (H, respectively) such that $g_i = c$, $h_j = d$ simultaneously for some i, j is impossible. Then $(G, H, \{g_i\}, \{h_j\})$ is called a G, H-structure on $T_m \times T_n$. Let $M(G, H, \{g_i\}, \{h_j\})$ be the sum of degrees of points $g_1, \ldots, g_s$ in G and points $h_1, \ldots, h_t$ in H. If $g_i \neq c$ for each i ($h_j \neq d$ for each j), then we add to M degree of c in G (d in H, respectively).

Obviously, the degree of each point $g_i \neq c$ in G ($h_j \neq d$ in H, respectively) is either 0, 1, or 2. The center c of T_m (d of T_n) has degree m (n) and each other point g_i (h_j) has degree 2 in T_m (T_n, respectively). Hence if $g_i \neq c$, $h_j \neq d$ for each i, j, then $M(T_m, T_n, \{g_i\}, \{h_j\}) = m + n + 2(s + t)$. In the opposite case, $M(T_m, T_n, \{g_i\}, \{h_j\}) = m + n + 2(s + t - 1)$.

Proposition 3.1. *Let $T_m \times T_n$ have a G, H-structure $(G, H, \{g_i\}, \{h_j\})$. Let $K \subset T_m \times T_n$ be a finite tree. Suppose that all awful vertices of K lie in Γ. Let R be the set of vertices in K containing all bad vertices of K. Suppose that R is split into two sets $\{a_1, \ldots, a_s\}$ and $\{b_1, \ldots, b_t\}$ such that*

$$(K, a_1, \ldots, a_s, b_1, \ldots, b_t) \subset_b (G \times H, g_1 \times d, \ldots, g_s \times d, c \times h_1, \ldots, c \times h_t).$$

Let $N = s + t$ be the number of vertices in R. Further we assume the defect of K is calculated over all vertices from R (not only bad). Then the following conditions hold:

(3.1.1) $\delta(K) + 2N \leqslant M(G, H, \{g_i\}, \{h_j\})$;

(3.1.2) *if vertices $a_1, \ldots, a_s, b_1, \ldots, b_t$ have no hanging edges, then*

$$\delta(K) + 2N < M\big(G, H, \{g_i\}, \{h_j\}\big),$$

hence if $\delta(K) + 2N = M$, then there is a vertex from R with a hanging edge;

(3.1.3) *if $\delta(K) + 2N = M(G, H, \{g_i\}, \{h_j\}) > 0$, then there exists a compression arc $A \subset K$ containing a vertex from R.*

Reduction Lemma 3.2. *Let $K \subset T_m \times T_n$ be a finite tree, $X \subset T_m$, $Y \subset T_n$ be subpolyhedra. If $K \subset_b X \times Y$, then there exist compressions $q_1, \ldots, q_k$ such that all awful vertices of $K' = q_k(\ldots(q_1(K))\ldots)$ lie in Γ and $\delta(K') \geqslant \delta(K)$.*

Proof of necessity in Theorems 1.1 and 1.4. Suppose that all awful vertices of K lie in Γ (simple case).

Case 1 (*the partial case* $m = 2$). For presenting the main ideas, we first prove the simple case for $m = 2$. Then necessity in Theorem 1.1 follows from Propositions (3.1.1) and (3.1.2) for $G = T_2$, $H = T_n$ as follows. Let R be the set of all bad vertices in K. Let $g_1 \times d, \ldots, g_s \times d$ be the images of all bad vertices of K under the given basic embedding $K \subset_b \mathbb{R} \times T_n$. Since $M(\mathbb{R}, T_n, \{g_i\}) = n + 2s$, $N = s$, then by (3.1.1) and (3.1.2) either $\delta(K) < n$ or $\delta(K) = n$ and K has a dry vertex.

Now we prove the simple case for $m, n \geqslant 3$. Let $g_1 \times d, \dots, g_s \times d, c \times h_1, \dots, c \times h_t$ be the images of all bad vertices of K under the given basic embedding $K \subset_b T_m \times T_n$. Since $M(T_m, T_n, \{g_i\}, \{h_j\}) \leqslant m + n + 2N$, then by (3.1.1) $\delta(K) \leqslant m + n$.

Case 2 (*$c \times d$ corresponds to a bad vertex*). Suppose there is a bad vertex $r = c \times d$ of K. Consequently, either $g_i = c, a_i = r$ for some i or $h_j = d, b_j = r$ for some j. Hence $M(T_m, T_n, \{g_i\}, \{h_j\}) = m + n + 2(N-1)$, i.e., (1.4.1) holds by (3.1.1) and (3.1.2).

Case 3 (*$c \times d$ does not correspond to a bad vertex*). Let $a_1, \dots, a_k$ and $b_1 \dots, b_l$ be all bad vertices of K, images of which lie in $(T_m - c) \times d$ and $c \times (T_n - d)$, respectively. Evidently, there are stars $T_{\deg a_1} \coprod \cdots \coprod T_{\deg a_k} \subset K$ basically embedded into m 'books' $(C' - c) \times T_n$, where C' is a hanging edge of T_m. By definition we have $M(T_m - c, T_n, \{g_i\}, \emptyset) = n + 2s$ and by (3.1.1) for $T_{\deg a_1} \cup \cdots \cup T_{\deg a_k} \subset (T_m - c) \times T_n$

$$(\deg a_1 - 2) + \cdots + (\deg a_k - 2) \leqslant n.$$

Moreover, by (3.1.2) when the equality holds, one vertex from $\{a_i\}$ (let it be a_1) is a dry vertex. Analogously we have

$$(\deg b_1 - 2) + \cdots + (\deg b_l - 2) \leqslant m$$

and, when the equality holds, b_1 is a dry vertex. So, (1.4.2) holds.

Case 4 (*general case*). In the general case (when not all awful vertices of K lie in Γ) by Reduction Lemma 3.2 there exist compressions $q_1, \dots, q_k$ such that all awful vertices of $K' = q_k(\dots(q_1(K))\dots)$ lie on Γ and $\delta(K') \geqslant \delta(K)$. Then necessity in Theorems 1.1 and 1.4 follows from the simple case for K'. $\square$

Proof of (3.1.1) and (3.1.3). Further, we briefly denote $M(G, H, \{g_i\}, \{h_j\})$ by M. Induction on M. Base $M = 0$ in (3.1.1): R is the set of $2N$ isolated points. Hence $\delta(K) = -2N$. Base $M = 1$ in (3.1.3): vertices from R have not more than one edge in K. Hence $\delta(K) \leqslant 1 - 2N$. The inductive step is Induction Lemma 3.3 below. $\square$

Induction Lemma 3.3. *Under the conditions of Proposition* 3.1 *we have that there exist a subgraph $L \subset K$ and an arc $B \subset L$ containing a vertex from R such that $\delta(L - \mathring{B}) = \delta(K) - 1$ and either for $G' = G - p_x\mathring{B}$, $H' = H$ or for $G' = G$, $H' = H - p_y\mathring{B}$ the following condition hold*:

(3.3.1) $(L - \mathring{B}, a_1, \dots, a_s, b_1, \dots, b_t) \subset_b (G' \times H', g_1 \times d, \dots, g_s \times d, c \times h_1, \dots, c \times h_t)$;

(3.3.2) $M(G', H', \{g_i\}, \{h_j\}) = M(G, H, \{g_i\}, \{h_j\}) - 1$.

Proof. By GC 2.1 there exists a maximal n for which $L = E^n(K) \cup R$ contains a neighborhood of every point from R in K. Evidently, $\delta(L) = \delta(K)$. Then for some point $r \in R$, $E(L)$ does not contain any neighborhood of r in K. So, there exists an edge of K with end r, say A, and a sequence $\{r_i\} \in A - E(L)$ converging to r. By definition of E we have either $L \cap (p_x r_i \times T_n) = r_i$ or $L \cap (T_m \times p_y r_i) = r_i$ for each i. We may assume that $L \cap (p_x r_i \times T_n) = r_i$ for each i. Since $E(L)$ is a finite graph, then $E(L)$ contains a finite number of connected components. Then $E(L)$ is split by graphs $p_x r_i \times T_n$ into a finite number of connected components. Hence there exists a subarc $B \subset A$ containing r such that $L \cap (p_x B \times T_n) = B$. So, (3.3.1) holds for $G' = G - p_x \mathring{B}$ and $H' = H$. Since the arc B contains the vertex $r \in R$, then (3.3.2) holds and $\delta(L - \mathring{B}) = \delta(L) - 1 = \delta(K) - 1$. □

Proof of (3.1.2). We shall prove (3.1.2) by induction on M (see Proposition 3.1). Bases $M = 1$, $M = 2$ in (3.1.2) are obvious (see the bases in the proof of (3.1.1) and (3.1.3)). Suppose to the contrary that $\delta(K) + 2N = M$. Then by (3.1.3) for K there is an inclusion maximal compression arc I_1 with endpoints $r \in R$ and $a \in K$. Take the compression q_1 generated by I_1. By (2.3.1) we have $q_1 K \subset_b (G/I_1) \times H$. By (2.3.2) only the following cases are possible:

(1) $a \notin R$ is a vertex of K;
(2) $a \in R$;
(3) $q_1 K \cong K$.

Case 1. In the first case, since a is non-hanging, then $\deg q_1 r$ in $q_1 K$ is greater than $\deg r$ in K. Hence $\delta(q_1 K) > \delta(K)$. Also the number of all vertices in R for $q_1 K$ equals N and

$$M\big(G/I_1, H, \{q_1 g_i\}, \{q_1 h_j\}\big) \leqslant M\big(G, H, \{g_i\}, \{h_j\}\big).$$

Then $\delta(q_1 K) + 2N > M$ (here M is for the basic embedding $q_1 K \subset (G/I_1) \times H$), contradicting (3.1.1).

Case 2. In the second case, since $q_1 r = q_1 a$, then the number of all vertices in R for $q_1 K$ equals $N - 1$ and

$$M\big(G/I_1, H, \{q_1 g_i\}_{g_i \neq a}, \{q_1 h_j\}_{h_j \neq a}\big) = M\big(G, H, \{g_i\}, \{h_j\}\big) - 2.$$

Since $(\deg r - 2) + (\deg a - 2) = (\deg q_1 r - 2)$, then we have $\delta(q_1 K) = \delta(K)$. Then $\delta(q_1 K) + 2(N - 1) = M$ (here M is for the basic embedding $q_1 K \subset (G/I_1) \times H$) and (3.1.2) follows from the inductive hypothesis.

Case 3. In the third case $\delta(q_1 K) = \delta(K)$. Note that we proved that the defect of a tree after a compression is not less than that at the beginning. So we may apply analogous compressions $q_2, \dots, q_k$, generated by arcs $I_2, \dots, I_k$, respectively. It suffices to prove that this process is finite.

Suppose there is a compression (let it be q_1) generated by I_1 at $r_1 \in R$ such that $q_1 K$ contains a compression arc I_2 at $r_2 \in R$ appearing due to q_1, i.e., I_1, I_2 are orthogonal and if I_1 is horizontal (vertical), then

$$r_2 \in p_x r_1 \times T_n, \qquad q_1^{-1}(I_2) \subset (p_x I_1) \times T_n$$
$$(r_2 \in T_m \times p_y r_1, \quad q_1^{-1}(I_2) \subset T_m \times (p_y I_1), \text{ respectively}).$$

After that, suppose there is an analogous arc $I_3 \subset q_2 K$, and so on. If we find such arcs $I_1, \dots, I_k$, then we may construct an array of $k + 1$ points in K as follows.

We may assume I_k is horizontal. Take a point $b_k \in I_k - r_k$. Then b_k, r_k is the array of 2 points in $q_{k-1} K$. Since I_k appears due to q_{k-1}, then there is $b_{k-1} \in I_{k-1} \cap (T_m \times p_y(q_{k-1}^{-1} b_k))$. Then $q_{k-1}^{-1} b_k, b_{k-1}, r_{k-1}$ is the array of three points in $q_{k-2} K$, and so on.

Since the map q_i^{-1} preserves the orthogonality of arcs, then we find the array of $k+1$ points in K:

$$\{q_1^{-1}(\ldots(q_{k-1}^{-1}(b_k))\ldots),\ldots,q_1^{-1}(b_2),b_1,r_1\}.$$

But there are no arrays of arbitrary length in K. Hence there is a constant C such that the length of the above constructed sequence of arcs $I_1,\ldots,I_k$ is less than C. Since the number of vertices from R is N, then there are not more than $N(m+n)$ compression arcs in K. Hence we can do not more than $(N(m+n))^C$ compressions, i.e., our process is finite. □

Proof of Reduction Lemma 3.2. Let r be an awful vertex of K such that $r \notin \Gamma$. Let C be the inclusion maximal cross in K with center r. Apply compressions $q_1,\ldots,q_k$ to C, analogous to the proof of (3.1.2). We have either a contradiction or $q_k(\ldots(q_1(r))\ldots) \in \Gamma$ for some k. We may iterate this procedure to each awful vertex of K that does not lie in Γ. And also, the defect of a tree after these compressions is not less than that at the beginning (see the remark in Case 3 of the proof of (3.1.2)). □

4. Construction of an admissible tree

This section is organized as follows. First we construct a pre-loaded leaf. After that we define a loaded leaf using a filtration of pre-loaded leaves. Finally, we construct simple and complete admissible trees using a filtration of loaded leaves.

The following construction is simplified for Theorem 1.1. In this case we do not split satisfactory points into horizontal and vertical. In particular, in the definition of a loaded leaf we omit condition (4.2.2b). Hence we also omit the notion of the end of the loaded leaf and the order of satisfactory points in the loaded leaf. Finally, we may alter the property Φ to the following: if $r_1,\ldots,r_k$ are all satisfactory points of a finite tree K, then $\phi(r_1)+\cdots+\phi(r_k) \leqslant n-1$. Remember that a tree basically embeddable into $\mathbb{R}^2$ contains only vertices either of degree $\leqslant 3$ or of degree 4 with a hanging edge.

Definition (a leaf and its root). Take a tree L basically embeddable into $\mathbb{R}^2$ with its endpoint r. Then L is called a leaf with the root r.

4.1. *Definition of a pre-loaded leaf*

Let I be a leaf. Take two of its hanging vertices $r,a \in I$ (i.e., endpoints) and two arbitrary sets of distinct points in the interior of edges of I (*good* and *satisfactory* points, respectively) such that these points lie in an arc $U \subset I$ with endpoints r and a (possibly $U=r=a$). Split the set of satisfactory points into *horizontal* and *vertical*. Moreover, we shall assume that r,a are satisfactory, and r is simultaneously both horizontal and vertical. For each satisfactory point $b \in I$ take an integer $\phi(b) \geqslant 0$ such that the property Φ below holds for $K=I$. Since each vertex of a leaf has either degree $\leqslant 3$ or degree 4 and a hanging edge, then, I is obtained from U by gluing to U either a hanging edge or a leaf, or both a hanging edge and a leaf at some points of U (called *excellent*). Then the tree I with its excellent, good and satisfactory points, and the function ϕ is called a *pre-loaded leaf*. The point r is called the *root* of I. The point a is called the *end* of I (possibly $I = r = a$). For example, in Fig. 4 the pre-loaded leaf with the root $u_0 = \alpha \times \beta$ and the end u_2 is represented by fat lines, q is the good point, t is the excellent point. If all satisfactory points of I are horizontal (vertical), then I is called *horizontal* (*vertical*, respectively).

Property Φ.

(a) *If $r_1 = r, r_2, \ldots, r_s$ are all distinct satisfactory points of a finite tree K, then*

$$\phi(r_1) + \cdots + \phi(r_s) \leqslant m + n - 3;$$

(b) *if $a_1, \ldots, a_k$ and $b_1, \ldots, b_l$ are all horizontal and vertical satisfactory points in $K - r$, respectively* (*for a horizontal and vertical pre-loaded leaf we have $l = 0$ and $k = 0$, respectively*), *then*

$$\phi(a_1) + \cdots + \phi(a_k) \leqslant n - 1, \quad \phi(b_1) + \cdots + \phi(b_l) \leqslant m - 1.$$

4.2. Definition of a loaded leaf

Let I_1 be a pre-loaded leaf. Let $I_1 \subset \cdots \subset I_k \subset J$ be a filtration such that the following conditions hold:

(4.2.1) I_{i+1} is obtained from I_i by gluing to I_i either a horizontal or a vertical pre-loaded leaf B and possibly a hanging edge H at each good vertex $b \in I_i - I_{i-1}$ ($I_0 = \emptyset$) for each $i = 1, \ldots, k-1$. Moreover, the root of B is the good point in I_{i+1} and also b is both an endpoint of H and the root of B; and either

(4.2.2a) J is obtained from I_k by gluing to I_k a leaf at the end of each pre-loaded leaf in I_k. Moreover, all satisfactory points of J_1 are horizontal (vertical). In this case J_1 is called *horizontal* (*vertical*, respectively); or

(4.2.2b) J is obtained from I_k by gluing to I_k a leaf at the end of each pre-loaded leaf in I_k, except one end a. In this case a is called the *end* of J_1. Moreover, all satisfactory points of J_1 before a (for the definition of the order, see below) and a itself are horizontal (vertical), and other satisfactory points of J_1 are vertical (horizontal, respectively).

Then the tree J with its excellent, good and satisfactory points, and the function ϕ such that the property Φ holds for $K = J$, is called a *loaded leaf*. Note that if a loaded leaf J has only one satisfactory point (obviously it is its root), then J is a leaf. Take a good point $b \in I_i$. Let B be the connected component of $J - b$ that is contained in $J - I_i$. Then the closure of B is the loaded leaf with the root b.

Definition (the order of satisfactory points in the loaded leaf). We shall define the order recursively. The order of satisfactory points in I_1 is from the root to the end along the arc U. The satisfactory points of I_1 in this order are the first satisfactory points in J. The order of good points in I_1 is from the end to the root along the arc U. The next points in J are satisfactory points in the loaded leaf beginning at the first good point in I_1 (in this loaded leaf the order is recursively defined), and so on. The last points in J are satisfactory points in the loaded leaf beginning at the last good point in I_1.

For example, in Fig. 4 the order of satisfactory points in the loaded leaf embedding into $A \times B$ is as follows: $u_0 = \alpha \times \beta, u_1, u_2, s$ (q_1, q_2 are not vertices of the loaded leaf).

Definition (a bridge and its ends in the loaded leaf). The subtree in J between two neighboring satisfactory points $b_1, b_2 \in I_1$ is called a bridge of J_1, and b_1 and b_2 are called the ends of the bridge.

Clearly, if a bridge B of J does not contain the end a of J_1, then all satisfactory points of B are either horizontal or vertical simultaneously.

4.3. *Definition of an admissible tree*

Let J_1 be a loaded leaf. Let $J_1 \subset \cdots \subset J_l = G$ be a filtration such that the following condition holds:

(4.3.1) J_{j+1} is obtained from J_j by gluing to J_j either $\phi(b)$ (if $b \neq r$ is not the end of a loaded leaf in J_j) or $\phi(b) + 1$ (if b either is an end of a loaded leaf in J_j or $b = r$) loaded leaves at each satisfactory point $b \in J_j - J_{j-1}$ ($J_0 = \emptyset$) for each $j = 1, \ldots, l-1$.

The tree G such that the property Φ holds for $K = G$ is called a *simple admissible* tree for $T_m \times T_n$, and the root of J_1 is called the *root* of G. Take a satisfactory point $h \in G$. K is obtained from G by gluing to G a hanging edge H at h such that h is an endpoint of H. The tree K is called a *complete admissible* tree for $T_m \times T_n$ ($m \geqslant 2,\ n \geqslant 3$). We shall say that a finite tree K is an *admissible* tree, if K is either simple or complete admissible.

5. Construction of a basic embedding

Theorem 5.1. *An admissible tree is basically embeddable into $T_m \times T_n$.*

Let c and d be the centers of T_m and T_n, respectively. Fix hanging edges C and D of T_m and T_n, respectively. Further, we assume that $c \times d$ is the lower left vertex of the square $C \times D$. In this section we shall construct a basic embedding of an admissible tree such that all horizontal (vertical) satisfactory points lie in $C \times d$ ($c \times D$, respectively).

Definition (operations X_ε and Y_ε). Fix a small $\varepsilon > 0$. Let C_ε (D_ε) be the ε-neighborhood of c in C (of d in D, respectively). For $Z \subset T_m \times T_n$ let

$$X_\varepsilon(Z) = \bigl\{z \in Z\colon \operatorname{card}\bigl(Z \cap (p_x z \times T_n)\bigr) > 1 \text{ and either } p_y z \in D_\varepsilon \text{ or } \operatorname{card}\bigl(Z \cap (T_m \times p_y z)\bigr) > 1\bigr\},$$

$$Y_\varepsilon(Z) = \bigl\{z \in Z\colon \operatorname{card}\bigl(Z \cap (T_m \times p_y z)\bigr) > 1 \text{ and either } p_x z \in C_\varepsilon \text{ or } \operatorname{card}\bigl(Z \cap (p_x z \times T_n)\bigr) > 1\bigr\}.$$

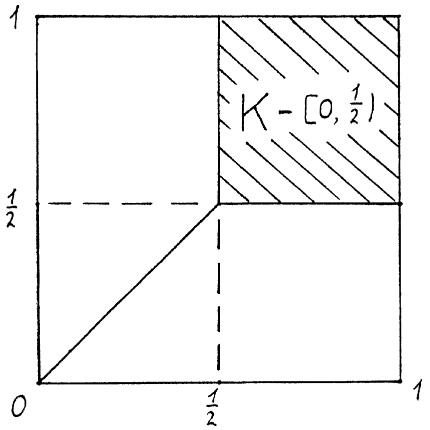

Fig. 3.

Definition (a strongly basic embedding). An embedding $Z \subset T_m \times T_n$ is called strongly basic (and denoted by $Z \subset_{sb} T_m \times T_n$), if there exist $\varepsilon > 0$ and an integer k such that $X_\varepsilon^k(Z) = Y_\varepsilon^k(Z) = \emptyset$. Then ε is called a *suitable* value for the strongly basic embedding.

Let us make the following remarks. Evidently, if Z is strongly basically embeddable into $T_m \times T_n$, then Z is basically embeddable into $T_m \times T_n$. Clearly, if $Z \subset T_m \times T_n$ and $X_\varepsilon^k(Z), Y_\varepsilon^k(Z)$ are strongly basic embedded into $T_m \times T_n$ for some k and ε, then Z is strongly basic embedded into $T_m \times T_n$. Obviously, if $X \subset Y$ and Y is strongly basic embedded into $T_m \times T_n$, then X is strongly basic embedded into $T_m \times T_n$. These statements shall be used in this section.

Now we shall present the scheme of our construction. Strongly basic embeddability of a simple admissible tree follows from Lemmas 5.3–5.5 below. In Lemma 5.3, using Proposition 5.2, we embed a horizontal loaded leaf. Evidently, Lemma 5.3 remains true if we replace the horizontal loaded leaf by a vertical loaded leaf. In Lemma 5.4 we embed a loaded leaf J with the end, assuming that all satisfactory points of J before the end of J and the end itself are horizontal, and the others are vertical (cf. (4.2.2b)). Obviously, Lemma 5.4 remains true if we replace all satisfactory horizontal points of J before the end of J and the end itself by vertical, others by horizontal. In Lemma 5.5 we extend an embedding constructed in Lemma 5.4 to a simple admissible tree. Basic embeddability of a complete admissible tree follows from these lemmas and Lemma 5.6. The following constructions are simplified for basic embeddings into $\mathbb{R} \times T_n$. In this case, strongly basic embeddability follows only from Lemma 5.3 and Steps 2, 3 in Lemma 5.5.

Proposition 5.2. *Let K be a leaf, $I \cong [0, 1]$ be its hanging edge* (*the vertex* 0 *is its root*). *Then there is a basic embedding*

$$\Big(K, K - \big[0, \tfrac{1}{2}\big), \big[0, \tfrac{1}{2}\big], 0\Big) \to \Big([0,1]^2, \big[\tfrac{1}{2}, 1\big]^2, \big[(0,0), \big(\tfrac{1}{2}, \tfrac{1}{2}\big)\big], (0,0)\Big) \quad (\textit{see Fig. } 3).$$

Proof. By [8, Theorem 1], $K - [0, \frac{1}{2})$ is basically embeddable into $\mathbb{R}^2$. It follows from [8, property F, p. 40] that there is a basic embedding

$$\left(K - \left[0, \tfrac{1}{2}\right), \tfrac{1}{2}\right) \to \left(\left[\tfrac{1}{2}, 1\right]^2, \left(\tfrac{1}{2}, \tfrac{1}{2}\right)\right).$$

The square $[\frac{1}{2}, 1]^2$ is called the *black square* of K. In Fig. 3 the black square is represented by the dashed square. □

Evidently, we may assume that there are both a hanging edge and a leaf at each excellent point of an admissible tree, and also that there is a hanging edge at each good point. Further, for an arbitrary set W, if $g : W \to T_m \times T_n$ is an embedding, then by 'W' we mean '$g(W) \subset T_m \times T_n$'. And also if $a, b \in K$ are two distinct points of a tree K, then by 'ab' we mean the arc in K with endpoints a, b.

Definition (the shadow). Let $W_1, W_2 \subset T_m \times T_n$ be two arbitrary sets. The shadow of W_1 is $(p_x W_1 \times T_n) \cup (T_m \times p_y W_1)$. The shadow of W_1 on W_2 is the intersection of the shadow of W_1 with W_2.

Lemma 5.3. *Let J be a tree basically embeddable into $\mathbb{R}^2$. Take a hanging vertex $r \in J$ and an arbitrary set of distinct points* (*called satisfactory*) *in the interior of edges of J. Then there is a strongly basic embedding $g : J \to [0, +\infty) \times [0, +\infty)$ such that $g(r) = 0 \times 0$ and all satisfactory points of J lie in $[0, +\infty) \times 0$.*

Proof. Evidently, we may find in the tree J a filtration satisfying conditions (4.2.1) and (4.2.2a). Hence we may assume that J is a horizontal loaded leaf (without a function φ satisfying Φ). The example of a strongly basic embedding is shown in Fig. 4, where we alter the quadrant $([0, +\infty) \times [0, +\infty), 0 \times 0)$ to the rectangle $(A \times B, \alpha \times \beta)$.

For simplicity, in Fig. 4 we do not show the hanging edges of excellent and good points. Dashed lines show some shadows of leaves and ε-neighborhoods for strongly basic embeddings. In Steps 1–3 below we embed J without leaves and hanging edges. The extension on leaves and hanging edges is constructed in Steps 4, 5, respectively. Fix a filtration $U \subset I_1 \subset \cdots \subset I_k \subset J$ from the definition of J (see Sections 4.1 and 4.2). Let $u_0 = r, u_1, \ldots, u_l$ be all satisfactory points of I_1 by the order from r.

Step 1 (*the 'decrease of the embedding' trick*). First we construct a strongly basic embedding $g : U \to A \times B$ using the following rules (see Fig. 4):

(5.3.1) $r = \alpha \times \beta, u_1, \ldots, u_l \in A \times \beta$;

(5.3.2) u_i lies in $A \times \beta$ to the right of u_{i-1}, $i = 1, \ldots, l$;

(5.3.3) projections under p_y of all excellent and good points of $u_{i-1}u_i$ lie in A higher than those of the arcs $u_i u_{i+1}, \ldots, u_{l-1}u_l$.

Step 2 (*the 'jump along the axis' trick*). Take the last good point $q \in u_{i-1}u_i \subset U$ by the order from r (if there is no such point, then we omit this step). Let J_1 be the connected component of $(J - U) \cup q$ containing q. Then J_1 is the horizontal loaded leaf with the root q. Split the hanging edge of q in J_1 into three parts by points q_1 and q_2. Extend g to $qq_2 = qq_1 \cup q_1q_2$ linearly so that

(5.3.4) $p_y q_1 \in p_y(u_{i-1}u_i)$ lies in A higher than $p_y q$;

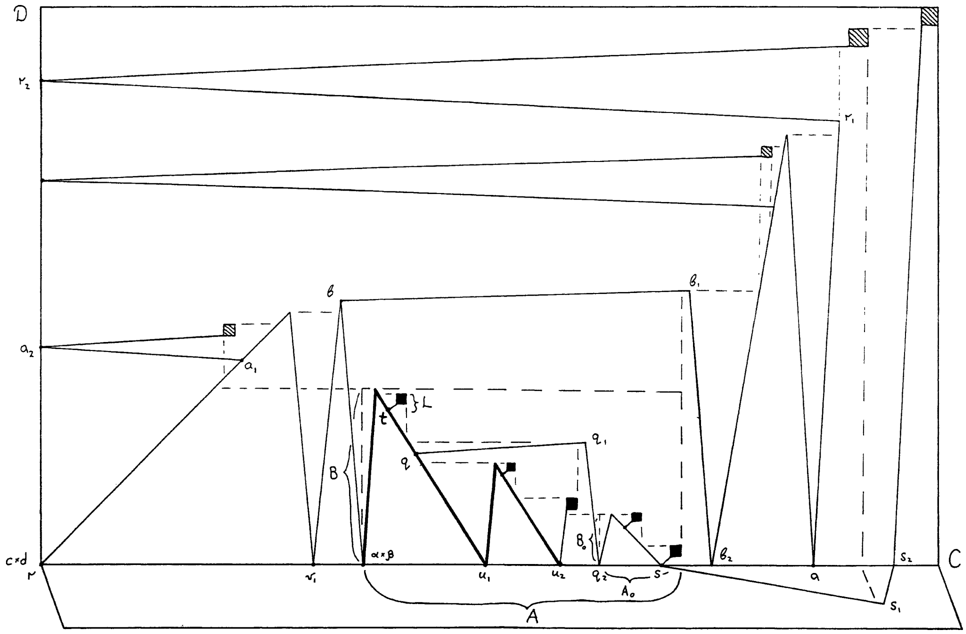

Fig. 4.

(5.3.5) $p_x q_1$ lies in $A \times \beta$ to the right of $p_x u$;
(5.3.6) q_2 lies in $A \times \beta$ to the right of $p_x q_1$;
(5.3.7) the shadow of $(qq_1 - q)$ on U and all excellent and good points of U are mutually disjoint.

Step 3 (*the 'ε-decrease of the embedding' trick*). Fix a suitable ε for the strongly basic embedding g on $U \cup qq_2$. Take two arcs $A_0 \subset A$ and $B_0 \subset q_2 \times B$ with the common endpoint q_2 so that

(5.3.8) $A_0 - q_2$ lies in $A \times \beta$ to the right of q_2, $p_y B_0 \subset B_\varepsilon$.

Obviously, we may assume q_2 is a satisfactory point and q_2 is the root of the loaded leaf $(J_1 - qq_2) \cup q_2$. Extend g to the arc $U_1 \subset (J_1 - qq_2) \cup q_2$ (the first pre-loaded leaf in the loaded leaf without leaves and hanging edges) analogous to Step 1. Further, as in Step 2 we take the last good point of U_1 in the order from q_2, and so on, until we embed the whole loaded leaf $(J_1 - qq_2) \cup q_2$ (without leaves and hanging edges). Evidently,

$$X_\varepsilon^3(U \cup J_1), Y_\varepsilon^3(U \cup J_1) \subset J_1 - qq_2 \subset_{sb} C \times D$$

for some ε. Hence g is a strongly basic embedding. Clearly, the maximal suitable ε for $g|_{U \cup J_1}$ is less than that for $g|_U$. After that, we analogously embed loaded leaves beginning at other good points of U. So, it remains to embed leaves at excellent points and hanging edges at excellent and good points of J.

Step 4 (*the embedding of leaves*). For a leaf L at the excellent point $t \in J$ we take a basic embedding $g : L \to A' \times B'$ from Proposition 5.2, where t is the common endpoint of arcs A', B' and such that

(5.3.9) all the shadows of leaves on J and all good points of J are mutually disjoint.

Since $g|_J$ is strongly basic (here J has no leaves and hanging edges), then for some k, ε we have that $X^k_\varepsilon(J)$ and $Y^k_\varepsilon(J)$ (here J has all leaves and has no hanging edges) consist of some leaves basically embedded into $A \times B$ such that its projections on $A \times \beta$ and $\alpha \times B$ are mutually disjoint. Hence the extension is strongly basic.

Step 5 (*the embedding of hanging edges*). Embed each hanging edge as a horizontal arc so that

(5.3.10) all the shadows on J of 'black squares' and hanging edges are mutually disjoint.

As in Step 4, the extension is strongly basic. □

Lemma 5.4. *Let J be a tree basically embeddable into $\mathbb{R}^2$. Take two hanging vertices $r, a \in J$* (*called the root and the end of J, respectively*). *And also take an arbitrary set of distinct points* (*called satisfactory*) *in the interior of edges of J such that all satisfactory points before a* (*see the definition of the order in Section* 4.2) *and a itself are horizontal, others are vertical. Then there is a strongly basic embedding $g : J \to C \times D$ such that $g(r) = c \times d$, all horizontal* (*vertical*) *satisfactory points of J lie in $C \times d$* ($c \times D$, *respectively*) *and a lies in $C \times d$ to the right of $p_x(J - a)$.*

Proof. Evidently, we may find in the tree J a filtration satisfying conditions (4.2.1) and (4.2.2b). Hence we may assume J is a loaded leaf with the end (without a function φ satisfying Φ). Further we embed J without leaves and hanging edges. The extension on leaves and hanging edges is constructed analogous to Steps 4, 5 in Lemma 5.3. Fix a filtration $I_1 \subset \cdots \subset I_k \subset J$ from the definition of J. Let $v_0 = r, v_1, \ldots, v_l$ be all satisfactory points of I_1 by the order from the root r. Let V_j be the bridge in J with endpoints v_{j-1}, v_j. Let the bridge V_{i+1} contain the end a (in Fig. 4, $i = 1$). If $a \neq v_{i+1}$, then we may assume there is only one vertex in V_{i+1}: a good point b such that the loaded leaf in $J - I_1$ beginning at b contains the end a. Actually, in the opposite case we take two points $v'_i, v'_{i+1} \in I_1$ near b (call them satisfactory, put $\phi(v'_i) = \phi(v'_{i+1}) = 0$) such that there is only one vertex b in the arc $v'_i v'_{i+1}$. Let V be the arc $rv_i \subset I_1$.

Step 1 (*the 'increase of the embedding' trick*) (cf. Step 1 in Lemma 5.3). First we shall construct a strongly basic embedding $g : V \to C \times D$. The case $V = r$ is obvious. We linearly define g on V using the following rules (see Fig. 4):

(5.4.1) $r = c \times d$, and $v_1, \ldots, v_i \in C \times d$;

(5.4.2) v_j lies in $C \times d$ to the right of v_{j-1}, $j = 1, \ldots, i$;

(5.4.3) projections under p_y of all excellent and good points of $v_j v_{j+1}$ lie in $c \times D$ higher than those of the arcs $rv_1, \ldots, v_{j-1}v_j$.

Step 2 (*the 'jump to the other axis' trick*) (cf. Step 2 in Lemma 5.3). Here we extend g to bridges $V_1, \ldots, V_i$. Evidently, all satisfactory points of the bridges, except their ends, are vertical. Take the first good point a_1 of rv_1 by the order from r. Let J_1 be the connected component of $(J - rv_1) \cup a_1$ containing a_1. Split the hanging edge of a_1 in J_1 into two parts by a point a_2. Then $(J_1 - a_1a_2) \cup a_2$ is the vertical loaded leaf beginning at a_2. Let ε

be a suitable value for the strongly basic embedding $g|_V$. Extend g to a_1a_2 linearly and to $(J_1 - a_1a_2) \cup a_2$ by Lemma 5.3 (for a vertical branch) so that

(5.4.4) a_2 lies in $c \times D$ higher than $p_y a_1$;

(5.4.5) $p_y(J_1 - a_1a_2) \subset p_y(rv_1)$ lies in $c \times D$ higher than a_2;

(5.4.6) $p_x(J_1 - a_1a_2) \subset C_\varepsilon$;

(5.4.7) the shadow of $J_1 - a_1a_2$ on rv_1 and all excellent and good points of rv_1 are mutually disjoint.

Evidently,

$$X_\varepsilon^2(V \cup J_1), Y_\varepsilon^2(V \cup J_1) \subset J_1 - a_1a_2 \subset_{sb} C \times D$$

for some ε. Hence g is a strongly basic embedding. Clearly, the maximal suitable ε for $g|_{V \cup J_1}$ is less than that for $g|_V$. After that, we analogously embed vertical loaded leaves beginning at other good points of V. So, we have now defined the strongly basic embedding on $V_0 = V_1 \cup \cdots \cup V_i$.

Step 3 (*the 'ε-decrease of the embedding' trick*) (cf. Step 3 in Lemma 5.3). If $a = v_{i+1}$, i.e., $i = l - 1$, then we extend g to V_{i+1} as in Step 2. After that the proof is finished. In the opposite case, extend g to $v_i v_{i+1}$ linearly using the following rule:

(5.4.8) $p_y b$ lies in $c \times D$ higher than $p_y V_0$, v_{i+1} lies in $C \times d$ to the right of $p_x b$.

Clearly, g on $V_0 \cup v_i v_{i+1}$ is strongly basic. Let J_0 be the connected component of $(J - v_i v_{i+1}) \cup v_{i+1}$ containing v_{i+1}. Evidently, J_0 is the horizontal loaded leaf. Let ε be a suitable value for the strongly basic embedding $g|_{V_0 \cup v_i v_{i+1}}$. Extend g to J_0 by Lemma 5.3 using the following rule (cf. (5.3.8)):

(5.4.9) $p_x J_0$ lies in $C \times d$ to the right of v_{i+1} and $p_y J_0 \subset D_\varepsilon$.

Since $g|_{V_0 \cup v_i v_{i+1}}$ is a strongly basic embedding, then there are ε, k such that

$$X_\varepsilon^k(V_0 \cup v_i v_{i+1} \cup J_0), Y_\varepsilon^k(V_0 \cup v_i v_{i+1} \cup J_0) \subset J_0 \subset_{sb} C \times D.$$

Hence the extension is strongly basic.

Step 4 (*the 'splitting of the embedding into layers' trick*). Suppose that $a \in I_j - I_{j-1}$ (j-layer) is contained in the loaded leaf P beginning at $b \in v_i v_{i+1}$ (in Fig. 4, $j = 2$). The proof is by induction on j. Base $j = 1$, i.e., $a = v_{i+1}$, was already proved. Inductive step. Split the hanging edge of b in P into three parts by points b_1 and b_2. Extend g to $bb_2 = bb_1 \cup b_1b_2$ linearly as in Step 3 of Lemma 5.3 (the 'jump along the axis' trick). Clearly, if $I'_1 \subset \cdots \subset I'_{j'}$ is a filtration for the loaded leaf P, then $a \in I'_{j-1} - I'_{j-2}$ ($(j-1)$-layer) is contained in the loaded leaf $(P - bb_2) \cup b_2$ beginning at $b_2 \in I_2$. By the inductive hypothesis there is an extension of g to $P - bb_2$ such that (see the 'increase of the embedding' trick)

(5.4.10) if ε is a suitable real for the strongly basic embedding g on $P - bb_2$, then $(J - P) \cup bb_2 \subset C_\varepsilon \times D_\varepsilon$.

Evidently, the embedding g is strongly basic. $\square$

Lemma 5.5. *Let G be a simple admissible tree. Suppose that there is a strongly basic embedding $g : J_1 \to C \times D$ such that $r = c \times d$, all horizontal* (*vertical*) *satisfactory points* *of J_1 lie in $C \times d$ ($c \times D$, respectively) and if J_1 has the end a, then a lies either to the right of or higher than $J_1 - a$. Then there is an extension*

$$g : G \to (T_m \times D) \cup (C \times T_n)$$

such that all horizontal (vertical) satisfactory points of G lie in $C \times d$ ($c \times D$, respectively).

Proof. *Step* 1 (*a loaded leaf at the end of the previous*). First suppose that J_1 has the end a. There is a strongly basic embedding

$$g : (J_1, a) \to \big(C' \times D', c' \times d'\big),$$

where C' and D' are subarcs of C and D containing c and d, respectively, and $c' \times d'$ is either the lower right or the upper left vertex of the square $C' \times D'$. Without loss of generality we may assume that a is the lower right vertex of the square. Take a loaded leaf $R \subset J_2 - J_1$ beginning at a. In the case when the second satisfactory point of R by the order from a is horizontal, extend g to R by Lemma 5.3 such that

(5.5.1) If ε is a suitable real for $g|_R$, then $J_1 \subset C_\varepsilon \times D_\varepsilon$.

Suppose that the second satisfactory point of R ordered from a is vertical. Split the hanging edge of a in R into three parts by points r_1 and r_2. Extend g to $ar_2 = ar_1 \cup r_1r_2$ linearly such that

(5.5.2) $p_x r_1$ lies in $C \times d$ to the right of a and $p_y r_1$ lies in $c \times D$ higher than $p_y J_1$;

(5.5.3) r_2 lies in $c \times D$ higher than $p_y r_1$.

Extend g to $R - ar_2$ by Lemma 5.3 (for a vertical branch) so that

(5.5.4) if ε is a suitable value for $g|_{R-ar_2}$, then $J_1 \cup ar_2 \subset C_\varepsilon \times D_\varepsilon$.

Note that after this step there are exactly $\phi(a)$ non-embedded loaded leaves of a in $J_2 - J_1$. If the loaded leaf R has the end, then we apply the previous to R instead of J_1, and so on, until we embed a subtree $W \subset G$ and the last embedded loaded leaf in W has no end.

Step 2 (*the 'choice of pages' trick*). Suppose that there is a non-embedded loaded leaf S of a satisfactory point $s \in W - r$. Without loss of generality we may assume that s is horizontal. Split the hanging edge of s in S into three parts by points s_1 and s_2. Since s is horizontal, then by Φ.(b) we may take a 'free page' of $C \times T_n$ (i.e., a 'page' $C \times D'$ not containing the already embedded subtree of G, where D' is a 'ray' of T_n). Linearly extend g to $ss_2 = ss_1 \cup s_1s_2$ using the following rules:

(5.5.5) $p_x s_1$ lies in $C \times d$ to the right of $p_x(W \cup ss_1)$;

(5.5.6) s_2 lies in $C \times d$ to the right of $p_x s_1$.

After that, extend g to $S - ss_2$ by Lemma 5.3 (cf. Step 1 of Lemma 5.4) so that

(5.5.7) if ε is a suitable value for $g|_{S-ss_1}$, then $W \cup ss_1 \subset C_\varepsilon \times D_\varepsilon$.

Evidently, the embedding g is strongly basic. Actually,

$$X_\varepsilon^2(W \cup S),\ Y_\varepsilon^2(W \cup S) \subset W \cup (S - ss_1) \subset_{sb} C \times D.$$

After that, we analogously embed other loaded leaves of G.

Step 3. Now it remains to embed only loaded leaves beginning at the root r of G. By the construction of G, the root r has $\phi(r) + 2$ loaded leaves in G. Clearly, we have already embedded exactly one of these loaded leaves into $C \times D$ (the loaded leaf J_1). First consider the partial case $m = 2$. Then, by property Φ we may embed first $\phi(r)$ loaded leaves at r into $C \times T_n$ the last branch at r into $(\mathbb{R} - C) \times D$ analogous to Step 2. In the general case Φ.(b) implies that we can find M and N such that $\phi(r) + 1 = M + N$ and

$$M + \phi(a_1) + \cdots + \phi(a_k) \leqslant n - 1, \qquad N + \phi(b_1) + \cdots + \phi(b_l) \leqslant m - 1,$$

where $a_1, \ldots, a_k$ and $b_1, \ldots, b_l$ are all horizontal and vertical satisfactory points of $G - r$, respectively. Thus, we may apply the 'choice of pages' trick as follows. First we embed M loaded leaves of r into 'free pages' of $C \times T_n$; the other N loaded leaves we embed into 'free pages' of $T_m \times D$. $\square$

Lemma 5.6. *Let K be a complete admissible tree and $G = (K - H) \cup h$ the respective simple admissible tree. Suppose that there is a strongly basic embedding*

$$g : G \to (C \times T_n) \cup (T_m \times D)$$

such that all satisfactory points of G lie either in $C \times d$ or in $c \times D$. Then there is a basic embedding

$$f : K \to (C \times T_n) \cup (T_m \times D)$$

such that $f|_G = g$.

Proof. Put $f|_G = g$. Since g is a strongly basic embedding, then there exist a real ε and an integer k such that $X^k_\varepsilon(G) = Y^k_\varepsilon(G) = \emptyset$. If $h \in C \times d$ $(c \times D)$ then we embed H into $C \times D$ as a vertical (horizontal) arc such that $p_y H \subset D_\varepsilon$ ($p_x H \subset C_\varepsilon$, respectively). Since $X^k_\varepsilon(K), Y^k_\varepsilon(K) \subset H$, then f is a basic embedding. $\square$

6. Proof of sufficiency in Theorem 1.4

Theorem 6.1. *A connected tree satisfying condition* (1.4.1) *is an admissible tree.*

Our aim is to select some filtrations in K satisfying conditions (4.2.1), (4.2.2), (4.3.1) and to call some vertices of K either excellent or good, or satisfactory, and to call each satisfactory point either horizontal or vertical such that the property Φ holds. In the partial case $m = 2$, the following constructions are simplified as follows. We may take an arbitrary root and Φ follows from $\delta(G) \leqslant n - 1$.

If $\delta(K) < m + n - 2$, then set $G = K$. In the opposite case, let G be K without a hanging edge at a dry vertex of K. Thus, $\delta(G) \leqslant m + n - 3$. So, it suffices to prove that G is a simple admissible tree. Call each bad vertex of K satisfactory. For each satisfactory point $b \in G$, set $\phi(b) = \deg b - 2$ in G. Then property Φ.(a) follows from $\delta(G) \leqslant m + n - 3$. Take a bad vertex $r \in G$ having the maximal number of leaves (let N) in G by comparison with other bad vertices of G. Call r both the root of G and the satisfactory point. If r is a unique bad vertex of G, then G is a wedge of leaves. Evidently, in this case property Φ.(b) holds, i.e., G is a simple admissible tree. In the opposite case, consider the closure A of a connected component of $G - a$, containing a bad vertex of G.

6.1. Selection of a pre-loaded leaf

We shall go along a path $U \subset A$ beginning at r, until we meet a vertex $b \in A$. Evidently, only the following cases are possible:

(1) b is a non-bad vertex, having either a leaf or a hanging edge or both a hanging edge and a leaf (call b an excellent point);

(2) b is a non-bad vertex without leaves and possibly having a hanging edge (call b a good point);

(3) b is a bad vertex (call b a satisfactory point).

In the first and second cases, we go along a non-passed edge of b in A. In the third case, we have either $\phi(b) \leqslant n-1$ or $\phi(b) \leqslant m-1$. Actually, in the opposite case (i.e., $\phi(b) \geqslant n$ and $\phi(b) \geqslant m$) let b have M leaves, i.e., there are $\deg b - M - 1$ non-passed connected components of $A - b$ containing a bad vertex. Evidently, for each such component B we have $\delta(B) \geqslant 1$. We obtain

$$m + n - 3 \geqslant \delta(G) \geqslant N - 1 + \phi(b) + (\deg b - M - 1).$$

Since $\phi(b) \geqslant m$ and $\deg b = \phi(b) + 2 \geqslant n + 2$, then

$$m + n - 3 \geqslant N - 1 + m + n + 2 - M - 1,$$

i.e., $M \geqslant N + 3$, that is contradicted by the choice of the root r.

Suppose that we already met satisfactory points $a_1, \dots, a_k$ (and called them horizontal) and $b_1, \dots, b_l$ (and called them vertical), and b is not in these lists. Set

$$\delta_x = \phi(a_1) + \cdots + \phi(a_k), \quad \delta_y = \phi(b_1) + \cdots + \phi(a_k).$$

At the very beginning $\delta_x = \delta_y = 0$. Then for a current meeting vertex b, we have either $\delta_x + \phi(b) \leqslant n - 1$ or $\delta_y + \phi(b) \leqslant m - 1$. The formal proof is analogous to that above: we alter the inequalities $\phi(b) \geqslant n$, $\phi(b) \geqslant m$ and

$$\delta(G) \geqslant N - 1 + \phi(b) + (\deg b - M - 1)$$

on $\delta_x + \phi(b) \geqslant n$ and $\delta_y + \phi(b) \geqslant m$ and

$$\delta(G) \geqslant N - 1 + \delta_x + \delta_y + \phi(b) + (\deg b - M - 1),$$

respectively. Suppose the previous vertex was called horizontal. If $\delta_x + \phi(b) \geqslant n$, then we stop at the previous step. If $\delta_x + \phi(b) \leqslant n - 1$, then we call b horizontal. If b also has a leaf, then we stop. In the opposite case, we go along a non-passed edge of b. Thus, we construct U until we stop. Since $\delta_x \leqslant n - 1$ and $\delta_y \leqslant m - 1$, then Φ.(b) holds. I_1 is obtained from U by gluing to U hanging edges and leaves from A at each respective excellent point of U. Clearly, I_1 is a pre-loaded leaf with the root r.

6.2. Selection of a loaded leaf

Evidently, only the following cases are possible:

(1) The end a of I_1 has a leaf in A.

(2) The end a of I_1 has no leaves in A, i.e., if a is horizontal (vertical), then for the next vertex $b \in A$ we have $\delta_x + \phi(b) \geqslant n$ ($\delta_y + \phi(b) \geqslant m$, respectively).

In the first case, we proceed as in Section 6.1 to select a pre-loaded leaf of the last good point of I_1 by the order from r, and so on, until either we sort out all good points of A or we get the case (2). In the second case, without loss of generality we may assume that a is horizontal, i.e., $\delta_x + \phi(b) \geqslant n$. Then we select pre-loaded leaves analogous to the case (1) with the following alterations: we start at the first (not the last) good point of I_1 (by the order from r) having a pre-loaded leaf and we call all bad vertices vertical. Moreover, Φ.(b) holds. Actually, if we get $\delta_y + \phi(s) \geqslant m$ for a current satisfactory point $s \in A$, then we obtain

$$m + n - 3 \geqslant \delta(G) \geqslant \delta_x + \phi(b) + \delta_y + \phi(s) \geqslant m + n,$$

and that is a contradiction. I_2 is obtained from I_1 by gluing to I_1 hanging edges and pre-loaded leaves at respective good points in A, and so on. J_1 is obtained from I_k by gluing to I_k a leaf from A at each end (except b) of pre-loaded leaves in I_k. So, by definition (see Section 4.2) J_1 is a loaded leaf, r is the root, and b is the end.

6.3. Selection of a simple admissible tree

Now, as in Section 6.2, we select loaded leaves at satisfactory points of J. Hence Φ.(b) holds. J_2 is obtained from J_1 by gluing to J_1 either $\phi(b) = \deg b - 2$ (if $b \neq r$ is not the end of J_1) or $\phi(b) + 1 = \deg b - 1$ (if either b is the end of J_1 or $b = r$) respective loaded leaves from G at each satisfactory point $b \in J_1$, and so on, until we get $J_l = G$.

7. Proofs of Corollaries 1.2 and 1.3

Proof of Corollary 1.2. It follows from Theorem 1.1 that in the case $n = 3$, if a finite graph K is basically embeddable into $\mathbb{R} \times T_3$, then $\delta(K) < 3$ or $\delta(K) = 3$ and K has a dry vertex. Clearly, K does not contain any of the graphs of Fig. 1. Evidently, W_n satisfies conditions of Theorem 1.1 for each n. Hence W_n is basically embeddable into $\mathbb{R} \times T_3$ for each n. So, it suffices to prove that Corollary 1.2(a) implies Corollary 1.2(b). It follows from Corollary 1.2(a) that:

(1) all vertices of K have degree less than five or have degree five and a hanging edge;
(2) there are no two vertices of K either having degree five or having degree four and without hanging edges.

Take a vertex $a \in K$ of maximal degree. By F we denote the closure of a connected component of $K - a$. It follows from (2) that F is a leaf. Then by Lemma 7.1 below, F is contained in V_n for some n. It follows from (1) that $K \subset W_n$ for some n, i.e., Corollary 1.2(b) holds. □

Lemma 7.1. *A leaf F is contained in V_n for some n.*

Proof. Let G be a tree F after elimination of one hanging edge at every non-hanging vertex of K (if this edge exists). Then by Lemma 7.2 below $G \subset U_n$ for some n. Hence, by construction of V_n, $F \subset V_n$ for some n. □

Lemma 7.2. *Let G be a finite tree. Suppose that all vertices of G have degree less than four. Then G is contained in U_n for some n.*

Proof. Let N be the number of all non-hanging vertices of G. Let us prove that there exists an embedding $G \subset U_N$ such that the root of U_N corresponds to a hanging vertex of G. Induction on N. Base $N = 1$ is obvious. To prove the inductive step, let A be a hanging edge of G with the non-hanging endpoint a. Then to A assign an edge B of $U_1 = T_3$ such that the center b of T_3 corresponds to a. Since $\deg a < 4$, then there are at most two connected components (denote its closures by H_1 and H_2) of $G - A$. The number of all non-hanging vertices for H_1 and H_2 is less than that for G. Moreover, by construction of U_N, the closures of two connected components of $U_N - B$ are two copies of U_{N-1}. Then, by the inductive hypothesis, there exist embeddings $H_1 \subset U_{N-1}$, $H_2 \subset U_{N-1}$ such that roots of two copies U_{N-1} correspond to a. So, we obtain an embedding $G = A \cup H_1 \cup H_2 \subset B \cup U_{N-1} \cup U_{N-1} = U_N$. □

Proof of Corollary 1.3. Consider the set of finite trees K such that either $\delta(K) > n$ or $\delta(K) = n$ and K has no dry vertices; and also $\delta(K) \leqslant 2n$. From these trees, choose minimal by inclusion trees and call them *prohibited* for $\mathbb{R} \times T_n$. It follows from Lemma 7.3 that there are only a finite number of prohibited trees. So, it suffices to prove that a finite graph K is basically embeddable into $\mathbb{R} \times T_n$ if and only if K is a tree and K does not contain any of prohibited trees for $\mathbb{R} \times T_n$. Evidently, if K is basically embeddable into $\mathbb{R} \times T_n$, then by Theorem 1.1, K does not contain any prohibited trees.

Now suppose that K does not contain any prohibited trees and K is not basically embeddable into $\mathbb{R} \times T_n$. Hence $\delta(K) > 2n$. Without loss of generality we may assume that K is connected. For each bad vertex $r \in K$ we have $\deg r \leqslant n + 2$. Actually, in the opposite case K contains the prohibited tree T_{n+3}. Evidently, there exists a bad vertex $r \in K$ having only one connected component G of $K - r$ with a bad vertex of K. Let K_1 be the closure of G. Hence $K_1 \subset K$,

$$\delta(K) > \delta(K_1) = \delta(K) - (\deg r - 2) > n,$$

and we may apply the previous to K_1. In some step, we get $K_l \subset K$ and $n < \delta(K_l) \leqslant 2n$. Consequently, K contains one of the prohibited trees. This is a contradiction. □

Lemma 7.3. *For each integer $k \geqslant 1$ there are a finite number of minimal by inclusion trees K such that $\delta(K) \leqslant k$.*

Proof. It suffices to prove that there are a finite number of minimal by inclusion trees K with $\delta(K) = k$. Evidently, a minimal by inclusion tree is a union of some stars. Each bad vertex $b \in K$ contributes $\deg b - 2$ into $\delta(K) = k$. Evidently, there are a finite number of ways to split $\delta(K) = k$ into a sum of positive integers. Consequently, for each term $l \geqslant 2$

in $\delta(K) = k$ we may take a star with $l + 2$ rays, and also there are a finite number of ways to connect a finite number of stars to a finite tree K. So, Lemma 7.3 is proved. □

8. Conjectures

The first conjecture is the following criterion for basic embeddability into $T_m \times T_n$.

Conjecture 8.1. A finite (not necessarily connected) graph K is basically embeddable into $T_m \times T_n$ if and only if K is a tree and either (1.4.1) or (1.4.2) holds.

By Theorem 1.4, it suffices to prove that if for a finite tree K condition (1.4.2) holds, then K is basically embeddable into $T_m \times T_n$.

Analogous to the proof of Corollary 1.3, Conjecture 8.1 implies the following Conjecture 8.2. But, possibly Conjecture 8.2 can be proved independently of Conjecture 8.1.

Conjecture 8.2. There exists only a finite number of 'prohibited' subgraphs for basic embeddings into $T_m \times T_n$. Consequently, for a finite graph K there is an algorithm for checking whether K is basically embeddable into $T_m \times T_n$.

Now we shall formulate a conjecture for basic embeddability into $G \times \mathbb{R}$, where G is a finite connected tree. Let A be the set of all non-hanging vertices of G. Let R be the set of all bad vertices of a finite graph K. For a map $\chi : R \to A$, let

$$\delta_{\chi,a}(K) = \sum_{r \in R:\ \chi(r)=a} (\deg r - 2).$$

Conjecture 8.3. A finite (not necessarily connected) graph K is basically embeddable into $G \times \mathbb{R}$ if and only if K is a tree and there exists a map $\chi : R \to A$ such that for each $a \in A$ either $\delta_{\chi,a}(K) < \deg a$ or $\delta_{\chi,a}(K) = \deg a$ and there is a dry vertex $r \in R$ with $\chi(r) = a$.

The following conjecture is for basic embeddings into a cylinder $S \times \mathbb{R}$ and a torus $S \times S$.

Conjecture 8.4.

(a) A finite graph K is basically embeddable into $S \times \mathbb{R}$ if and only if K does not contain any of the graphs of Fig. 5;

(b) A finite graph K is basically embeddable into $S \times S$ if and only if K does not contain any of the graphs of Fig. 6.

Theorem 1.1 consists of two parts: a natural one involving the defect and an unnatural one involving horrid and awful vertices. One can conjecture that this theorem is a partial case of some combinatorial (not topological) one, involving defect but not involving horrid or awful vertices, just as the Kuratowski theorem and the Archedeacon–Hunecke

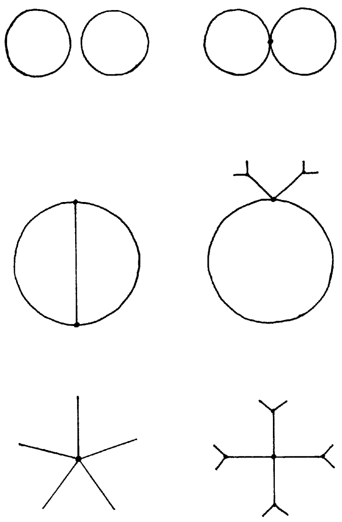
Fig. 5.

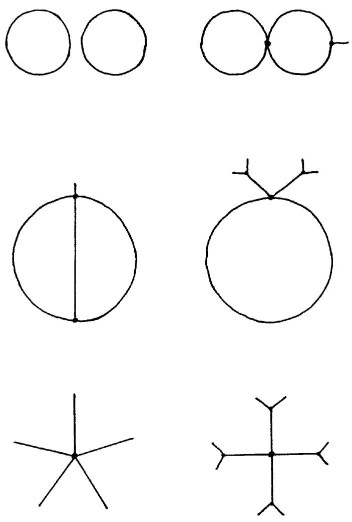
Fig. 6.

description of graphs embeddable into $\mathbb{R}^2$ and $\mathbb{R}P^2$ are partial cases of the Robertson–Seymor theorem on graph minors.

Conjecture 8.5. Suppose that A is a finite family of graphs with base points. Call a family M of graphs A-good if

(1) if $K \in M$, then every subgraph of K is in M;

(2) if $K \in M$, $x \in K$ and the closure L of a connected component of $K - x$ does not contain (topologically) subgraphs from the family A, then $(K/L) \in M$.

Then for each A-good family M there is a number N such that $K \in M$ if and only if the defect of K is less than N. The defect is the sum $\delta(K) = (\deg A_1 - 2) + \cdots + (\deg A_k - 2)$ over all vertices $A_1, \ldots, A_k$ of K that are base points of some subgraph $L \in A$ of K.

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